Equations of Geostrophic Flow
Due Monday 3 March 2003

In the last assignment we studied spherical coordinates and bases. We discovered that any 3D vector $\mathbf{v}$ can be expanded in terms of the three basis vectors $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ and $\mathbf{e}_\rho$. In this project we will continue with this line and derive the so-called **geostrophic equations of motion** in a spherical coordinate system. These equations are fundamental to understanding basic geophysical fluid phenomena such as tides, currents, waves as well as macroscopic properties of hurricanes and jets.

1. Write a summary of the "Bases and Spherical Coordinates" assignment. Your summary should include the formulas for the basis vectors and the formula for $\mathbf{\Omega}$ and its coordinates in terms of the spherical basis.

2. The **Coriolis acceleration f**, the contribution that the rotation of the planet makes to accelerate every fluid particle in the oceans, is given by the formula

$$\mathbf{f} = 2\mathbf{\Omega} \times \mathbf{v} \tag{1}$$

   where $\mathbf{v}$ is the velocity of the fluid particle relative to our planet and $\mathbf{\Omega}$ is the rotation rate of the Earth.

   Let $(\frac{\pi}{6}, \frac{\pi}{4})$ be the longitude and latitude of a specific position P on our planet where a particle of fluid is located and is moving with velocity $\mathbf{v}$.

   (a) Write down the three basis vectors $\{\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_\rho\}$ at this point.

   (b) Determine the components of $\mathbf{\Omega}$ in terms of the above basis.

   (c) Let the fluid particle located at P have speed 3 m/s in the northeast direction. Determine the components of $\mathbf{v}$ in the directions of the basis vectors, i.e., find v_1, v_2 and v_3 such that

$$\mathbf{v} = v_1\mathbf{e}_\theta + v_2\mathbf{e}_\phi + v_3\mathbf{e}_\rho. \tag{2}$$

   (d) Use your knowledge of cross product and determine $\mathbf{f}$ in this specific setting.

3. Generalize the results of the previous problem to any point on our planet, i.e., let (θ, ϕ) be the longitude-latitude of a point on our planet with a fluid particle located there and moving with velocity $\mathbf{v} = v_1\mathbf{e}_\theta + v_2\mathbf{e}_\phi + v_3\mathbf{e}_\rho$. Compute the components of $\mathbf{f}$ in this setting. (Ans: $\mathbf{f} = f_1\mathbf{e}_\theta + f_2\mathbf{e}_\phi + f_3\mathbf{e}_\rho$ where $f_1 = 2\Omega(v_3\cos\phi - v_2\sin\phi)$, $f_2 = 2\Omega v_1 \sin\phi$ and $f_3 = -2\Omega v_1\cos\phi$.)

4. **Equations of Geostrophic Motion**: In a geostrophic motion, by definition, three forces balance each other out to create an environment of motion for the fluid particles. These forces are due to **pressure gradient**, $-\nabla p$ to be exact, the force due to Coriolis, namely $\rho_0\mathbf{f}$, and the weight of the fluid, which will be directed to the center of the planet and therefore has the form $-\rho_0 g\mathbf{e}_\rho$, where ρ_0 is the density of the fluid and g is the constant of gravitational acceleration. Putting these expressions together, the equations of geostrophic flow become

$$\rho_0\mathbf{f} = -\nabla p - \rho_0 g\mathbf{e}_\rho. \tag{3}$$

We now make an important assumption, often referred to as the β-plane assumption, that nearby to a point P we can ignore the curvature of our planet and replace the sphere by its tangent plane. This means that we can replace our coordinates curves for θ and ϕ by the less intimidating x and y (cartesian) axes. So, for instance, we can write

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{e}_\theta + \frac{\partial p}{\partial y}\mathbf{e}_\phi + \frac{\partial p}{\partial z}\mathbf{e}_\rho.$$

Use the result from 3) regarding to show that equation (3) reduces to

$$\begin{cases} 2\rho_0\Omega(v_3\cos\phi - v_2\sin\phi) &= -\frac{\partial p}{\partial x}, \\ 2\rho_0\Omega v_1\sin\phi &= -\frac{\partial p}{\partial y}, \\ -2\rho_0\Omega v_1\cos\phi &= -\frac{\partial p}{\partial z} - \rho_0 g. \end{cases} \tag{4}$$

5. **2D Approximation**: Let's consider flows whose prominent variations are in the east-west (or x) and north-south (or y) directions. This assumption allows us to set $v_3 = 0$ as well as ignore the left-hand side of (4)c in relation to g. Thus (4)c reduces to

$$\frac{\partial p}{\partial z} = -\rho_0 g. \tag{5}$$

Assuming that the density ρ_0 is constant, show that (5) implies

$$p(x,y,z) = p_0(x,y) - \rho g z \tag{6}$$

where p_0 is the pressure at $z = 0$. Expression (6) is referred to as the **hydrostatic pressure** and states that the pressure measured at any point (x,y,z) is due only to the weight of the column of fluid above it.

6. The remaining two equations in (4) now take the form (recall that $v_3 = 0$)

$$-\rho_0 f v_2 = -\frac{\partial p}{\partial x}, \quad \rho_0 f v_1 = -\frac{\partial p}{\partial y}, \tag{7}$$

where $f = 2\Omega\sin\phi$ is called the **Coriolis parameter**.

   (a) Estimate the value of f at 30 degrees latitude.

   (b) Determine if this flow is incompressible, assuming that ρ_0 and f are constant.

   (c) Compute the vorticity of this flow as a function of pressure, assuming that both f and ρ_0 are constant.

   (d) What partial differential equation must p satisfy for the geostrophic flow to be irrotational?

7. Use (7) to show that isobars coincide with particle paths of $\mathbf{v}$ defined by (7). (Hint: Recall two facts about a) the geometric relation between particle paths, their tangents and $\mathbf{v}$, and b) the relationship between ∇p and an isobar.)

8. Let P, a point located in the northern hemisphere, be the center of a low pressure field in a 2D-geostrophic flow. Determine whether the motion around this pressure field is clockwise or counterclockwise. (Hint: First determine the sign of f at P. Next, recall the relationship between ∇p and an isobar. Use the fact that we have a *low* pressure field to decide which direction ∇p should face at a point near P. Finally, use (7) to infer the direction of $\mathbf{v}$.)